Size Matters - “Over”investments in a Relational Contracting Setting

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November 29, 2016

Abstract

The corporate finance literature documents that managers tend to overinvest in their companies. A number of theoretical contributions have aimed at explaining this stylized fact, most of them focusing on a fundamental agency problem between shareholders and managers. The present paper shows that over-investments are not necessarily the (negative) consequence of agency problems between shareholders and managers, but instead might be a second-best optimal response if the scope of court-enforceable contracts is limited. In such an environment a firm has to rely on relational contracts to manage the labor relations with its workforce. The paper shows that investments into physical productive assets enhance the enforceability of relational contracts and hence investments are “too high” optimally.

JEL Codes: C73, D21, D86, G32

Keywords: relational contracts, corporate finance, capital investments

*We thank Dan Barron, Sean Chu, Oscar Contreras, Wouter Dessein, Bob Gibbons, Sebastian Kranz, Jin Li, James Malcomson, Stefanos Mouzas, Mike Powell, Heiner Schumacher, Carmit Segal, Caspar Siegert and participants at the 2012 IIOC (Washington), the 2012 Conference of the German Economics Association (Göttingen), the 2013 European Meeting of the Econometric Society (Gothenborg) and the 2016 SIOE meeting (Paris) for helpful comments.

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1 Introduction

A prominent and well established stylized fact in corporate finance is that managers tend to overinvest, or, as Stein (2003) puts it, that they “...have an excessive taste for running large firms, as opposed to simply profitable ones” (p. 119). Numerous theories have been developed to explain this pattern: managers’ taste for empire building; see Williamson (1964), Jensen (1986), Jensen (1993); short-termism of managers who focus on activities the market can easily observe; see Stein (1989), Bebchuk and Stole (1993); managerial overconfidence into their own abilities; see Roll (1986), Heaton (2002); or asymmetric information with respect to new investment opportunities; see Inderst and Klein (2007). All these theories share the perception that over-investments are caused by agency problems between a firm and its management. Hence, mechanisms to reduce free cash-flow – and consequently the management’s ability to invest – have been suggested as optimal responses to this perceived fundamental agency problem. However, there is no clear evidence that reducing a firm’s free cash-flow and restricting a manager’s investment opportunities increases firm value - on the contrary, investors often assess capital investments positively; see McConnell and Muscarella (1985), Myers (2003).

This paper shows that over-investments – investments where marginal costs exceed (direct) marginal benefits – can have a positive impact on firm value, namely by making it easier to motivate a firm’s workforce: If formal, court-enforceable incentive contracts are limited in their scope – e.g., because of a lack of good verifiable performance measures – managing a firm’s employer-worker relationships in a volatile environment becomes difficult. In this case, implicit agreements (so-called relational contracts) are needed to motivate a firm’s workforce. If the firm’s commitment in these relational contracts is limited, though, a firm’s employment might be restricted to a level that is inefficiently low because the firm cannot make credible promises to all agents it would like to motivate. Moreover, if it is unable to motivate the employee, it is not optimal to employ him in the first place. Similar problems can arise if firms are liquidity constrained in some states of the world which means that they have to defer payments into the future, exacerbating commitment problems.

Then, managers optimally overinvest in order to improve the enforceability of relational contracts with their employees as capital investments ceteris paribus increase future revenues, whereas the associated investment costs are sunk. Thereby, larger investments A) strengthen the firm’s commitment by raising future profits, and B) increase its financial flexibility by generating additional cash flow. Hence, over-investments partially mitigate contracting frictions, and are not necessarily an inefficient manifestation of intra-firm agency problems.

The starting point of this paper is the moral hazard principal agent literature which
focuses on unobservable effort choice as a determinant of firm profitability (or productivity). An improved solution to the moral hazard problem will, ceteris paribus, increase productivity. While there exists a large literature, building on Holmstrom (1979) and Grossman and Hart (1983), focusing on explicit contracts that reward the agent based on verifiable performance measures, there has been an increased interest in implicit contracts as a way to mitigate the moral hazard problem; see, e.g., Bull (1987), MacLeod and Malcomson (1989), Levin (2003). In a more recent contribution, Gibbons and Henderson (2013) argue that different aspects of relational contracts are responsible for observed persistent performance differences among seemingly similar enterprises that also exist within industrialized countries.

Relational (or implicit) contracts employ repeated game logic to use observable but unverifiable information and do not rely on explicit, court enforceable, performance measures to motivate workers. In the recent past there have been a number of papers investigating richer dynamics and the effect of stochastic shocks on the efficiency and stability of implicit contracts – see, e.g., Li and Matouschek (2013), Englmaier and Segal (2011) – to which the present paper relates. Finally, we relate to some recent papers linking a firm’s financing conditions and decisions to the enforceability of relational contracts. Contreras (2013) analyzes how relational contracts formed between a firm and its supplier interact with the quality of financial markets. Fahn, Merlo, and Wamser (2014) show how equity financing helps to enforce relational contracts. Debt increases a firm’s reneging temptation because some of the negative consequences of breaking implicit promises can be shifted to creditors. In a related vein, Barron and Li (2015) explore how the negative effect of debt on the enforceability of relational contracts affects firm dynamics. They show that it is optimal for firms to first meet its financial obligations at the expense of having low compensation and effort levels in early periods.

In this paper, we develop a model where a firm’s inputs are physical capital and labor, and the time horizon is infinite. The firm can employ an arbitrary number of agents and output in every period is increasing in the firm’s workforce, as long as the employed agents choose high effort. Moreover, capital investments are made at the beginning of the game and increase output as well.

Different from the corporate finance literature – where over-investments generally are the consequence of agency problems between shareholders (who might have financed the initial investment into capital) and a firm’s management – we assume that both parties’ preferences are perfectly aligned. However, there are two features of the environment that can restrict the firm’s behavior and consequently reduce efficiency and profits:
First, neither an employee’s effort nor the resulting output are verifiable.\footnote{To motivate this presumption, assume that only very noisy performance measures for an employee’s effort are available and output is intangible.} As the firm and its management want to induce agents to exert high effort and agents have to be sufficiently incentivized, relational contracts where the firm implicitly promises to reward performance have to be employed. In case the firm does not honour its promise of rewarding performance, this will induce a loss of trust of the firm’s workforce, which will subsequently refrain from exerting high effort. Note that if the firm reneges on one of its promises, it is optimal to renge on all of its promises. Finally, hiring additional workers not only increases equilibrium profits but also the temptation to deviate. All this implies that the firm only has an incentive to honor its promises if the future value of the firm – determined by the net present value of its future profits – is sufficiently large. This constraint on credible promises concerning future rewards for high effort will limit the number of agents the firm can motivate (and hence profitably employ) through such promises. The firm may be restricted to an employment level that is below the (unconstrained) profit-maximizing value. Hence, frictions in the enforcement of formal contracts can restrict firm size, a result also derived by Powell (2016).

Second, we explicitly model the consequences of a volatile environment. In particular, we assume that the firm may face varying market conditions: demand may be either high or low. In the main part of the analysis, we assume that the firm faces a liquidity constraint and can only use generated cash-flow – funds that have been earned by selling its output – to compensate employees.\footnote{In a robustness exercise we show that the ability to accumulate and hold cash reserves does not solve this problem: Holding cash reserves is associated with present costs - these funds cannot be consumed today - in exchange for future profits, since higher employment can only be enforced in later periods. Hence, it will not be optimal to accumulate cash reserves that are sufficient to enforce efficient employment.} Under this assumption an employee’s compensation will have to vary with the company’s earnings. Then, a major part of total payments is paid out to agents whenever demand is high. However, the maximum amount the firm is willing to pay out instead of reneging and shutting down is determined by its expected future profits, independent of the difference between available liquidity across states of the world. A larger difference between the payoff in good and bad states and hence higher payment obligations in the good state increases the firm’s reneging temptation and will limit the firm’s employment to an inefficiently low level even if its discount factor is arbitrarily close to (but still bounded away from) 1.

The firm can mitigate these problems with motivating and managing its workforce by (seemingly excessive) investments into physical capital. Since investment costs are sunk, they are not considered by the firm whenever it faces the decision whether to keep its promises in the relational contract or not. Due to sunk investment costs and as an increased capital
base positively affects future profits, investments into capital improve the enforceability of relational contracts. In this context, over-investments – i.e., capital levels where the marginal investment costs exceed the direct marginal benefits – are possible. Over-investments also relax employment restrictions induced by low liquidity: additional investments raise the output in all states and hence increase the available cash-flow also in bad states of the world, i.e., when demand is low.

This result is interesting in comparison to the classic literature where over-investments are interpreted as a consequence of agency problems between shareholders and management: Means to reduce free cash-flow – like issuing debt – are proposed as (second-best) solutions to the problem of over-investments; see, e.g., Hart and Moore (1995), Zwiebel (1996). We show that the additional cash-flow generated by over-investments can be used to increase the firm’s financial flexibility and mitigate the agency problem between the firm and its workforce.

Our paper is related to Halac (2015). She analyzes a setting where a principal and an agent interact repeatedly, and the principal has to make an ex-ante relationship-specific investment. This classic hold-up problem induces the principal to under-invest if she fears that the returns from this investment will ex-post be expropriated by the agent. Then, the inability to use formal contracts may increase the efficiency of the relationship, by a logic similar to ours: A higher relationship-specific investment increases the benefits of keeping the relationship going, reduces the reneging temptation and thereby increases effort in the relational contract. Having to use relational contract may therefore mitigate the under-investment problem induced by hold up.

We invert this argument and state that over-investments might be optimal. In addition, we develop an additional benefit of over-investments. Those not only increase the value of the relationship, but also can help to deal with problems generated by volatile returns. In an extension where the liquidation value of the asset is positive, we further show that, different from Halac (2015), this mechanism does not (only) rely on the relationship-specificity of the investment: If the liquidation value of the asset increases (which implies that it becomes less relationship specific), the scope for over-investments might actually go up. Then, the principal’s outside option becomes more attractive, increasing her incentives to deviate, which increases the benefits of over-investments. On the contrary, the increase in the firm’s outside option is more pronounced for larger investments, which reduces the benefits of over-investments. If limited liability restricts enforceable employment, the former effect is more pronounced and more likely to dominate.

Moreover, our results are in line with several empirical studies comparing the behavior of private and publicly listed firms. The former are generally regarded to attach more impor-
tance to long-term relationships, which in our model translates into having a larger discount factor. We predict that the use of over-investments generally is more pronounced if the firm’s discount factor is small. This relates to Bargeron, Schlingemann, Stulz, and Zutter (2008), who show that publicly listed US firms pay substantially more for acquisitions than privately held firms. According to our model, such investments are more valuable for publicly listed than for privately-held firms – since the former are supposed to have lower discount factors and hence are more likely to be constrained by volatile returns. Having a larger firm allows to effectively use an internal capital market, thereby relaxing a firm’s limited liability constraint. Hence, the willingness to pay for acquisitions of publicly listed firms should ceteris paribus be higher. In addition, the “constrained-liquidity+limited-commitment” mechanism is most likely to restrict employment for intermediate discount factors. This also implies that residual profits (more precisely, freely available cash flows) vary more at firms with a higher discount factor, because constrained liquidity forces many of them to pass all their revenues on to employees if demand is low. Firms with a low discount factor, on the other hand, can also keep some cash in low-demand states. Given we expect firms to pay higher dividends to their shareholders in states where they have more free cash flow, we predict dividend payments of firms with a higher discount factor to vary more than of firms with a lower discount factor. This prediction is consistent with data documented by Michaely and Roberts (2012): privately held firms (having higher discount factors) smooth dividends significantly less than firms whose shares are publicly traded.

The paper proceeds as follows. After introducing the basic model setup and the firm’s objective in Sections 2 and 3, we show in Section 4 how the firm’s liquidity constraint can restrict firm size. If the volatility in the market, i.e., the difference between the firm’s earning in the good and bad state, is sufficiently large, employment always is inefficiently low, irrespective of the total expected surplus that can be generated. In Section 5, we show that investments into physical assets relax the firm’s constraints, which triggers over-investments whenever employment is restricted. In Section 6, we show that our results still hold when the asset has a positive outside value, when a credit market is available, and when the firm is able to retain earnings. In Section 7, we discuss implications of our results, shows how our results map into stylized facts, and presents an empirical prediction. The Appendix collects proofs and details of the derivations.
2 Model Setup

We first describe the basic model. In the next subsection we will describe the informational structure of the game. There is one firm ("principal"). Inputs for production are capital and labor. Time is discrete, the time horizon infinite, and all players are risk-neutral and share a common discount factor $\delta \in (0, 1)$. In the first period of the game, $t = 0$, the firm makes capital investments $k \in \mathbb{R}^+$. Capital investments are associated with marginal investment costs of 1. They can either be funded by the firm itself or raised from (passive) outside investors. To sharpen our argument, we abstract from any agency conflicts between the principal and outside investors. Hence, the following analysis does not depend on the sources of funds and we do not further pursue this issue. Furthermore, we assume that capital investments are relationship specific and for simplicity set the outside value of invested capital to zero (in an extension in Section 6.1, we allow the asset’s liquidation value to be positive).

The labor market consists of a mass $N$ of homogenous prospective employees (“agents”). In every period $t = 1, 2, \ldots$, the firm makes a short-term employment offer to $n_t \geq 0$ agents; if $n_t = 0$, the firm consumes its outside option $\pi \geq 0$. Agents receiving an offer are indexed $i \in [0, n_t]$. $N$ – the size of the labor market – is sufficiently large for the firm to not be exogenously bounded when choosing $n_t$. Each offer consists of a fixed wage $w_{it} \geq 0$ and the promise to make a contingent bonus payment $b_{it} \geq 0$. This bonus promise provides the agents with incentives and is paid at the principal’s discretion.

An agent’s decision whether to accept an offer or not is captured by $d_{it} \in \{0, 1\}$, where $d_{it} = 1$ describes an acceptance and $d_{it} = 0$ a rejection. The number of agents who accept an offer is $n_t$, i.e., $n_t = \int_0^{n_t} d_{it} di$. After accepting an offer, all $n_t$ employed agents then make their effort choice. Effort is binary, $e_{it} \in \{0, 1\}$, and associated with private costs $c(e_{it})$, where $c(0) = 0$ and $c(1) = c > 0$. All agents who rejected an offer as well as those who did not receive one consume their outside options $\overline{u} \geq 0$ in the respective period.

The firm’s output in period $t$ is $y_t = f \left( \int_0^{n_t} e_{it} di, k \right)$. Note that if all of the $n_t$ employed agents choose $e_t = 1$, then $y_t = f \left( n_t, k \right)$. $f (\cdot, k)$ is a continuous function in both arguments, with $f_1, f_2 > 0$ and $f_{11}, f_{22} < 0$. We do not place restrictions on the sign of $f_{12}$ but assume that the second order condition in the maximization problem below is always satisfied.

After producing, the firm sells the output and generates revenues $\theta_t y_t$. $\theta_t \in \{\theta^l, \theta^h\}$ is a parameter specifying the demand conditions – think “price” – for the firm’s output - with $0 < \theta^l < \theta^h$ - and is realized after the output has been produced. High demand is realized with probability $p$, low demand with $1 - p$. These probabilities are independent over time.

\footnote{Note that the restriction of payments to non-negative values is without loss of generality in our setting.}
i.e., there is no persistency in demand conditions. After the sale of output, payments $w_{it}$ and $b_{it}$ are made. Hence, the bonus can be contingent on the realization of $\theta$ (in addition to chosen effort), i.e., $b_{it}(e_{it},\theta_t)$, whereas $w_{it}$ is fixed by assumption.

We assume the firm to be liquidity constrained: all funds used to compensate employees must be earned via the sale of its products. This implicitly assumes that the firm does not retain profits earned in earlier periods and has no access to credit markets. We relax the first assumption in Section 6 and show that it does not have any impact on our qualitative results if retaining profits is costly. The second assumption is without loss of generality if firms can not use their physical assets to collateralize a loan (e.g., because they have zero value to outsiders). In this case borrowing would only shift the commitment problem from the firm’s relationship with its employees to the relationship with its creditors.

The timing within every period $t$ is summarized in the following graph:

Information, Payoffs, Strategies and Equilibrium

Generally, we assume that no contingent formal contracts are feasible. However, there are no informational asymmetries between players, hence agency problems only arise because formal contracts cannot be used to motivate employees\(^4\).

More precisely, a formal, court-enforceable contract can neither be based on an employed agent’s effort $e_{it}$, on output $y_t$, revenues $\theta_t y_t$, nor on the realization of the demand parameter $\theta$.

However, all these aspects, as well as employment levels $n_t$, acceptance decisions $d_{it}$, and wage and bonus payments can be observed by the principal as well as by all other agents. Thus, multilateral relational contracts can be used to motivate employees; see Levin (2002). Informally speaking, this implies that all agents as well as the labor market can detect any

\(^4\)See Englmaier and Segal (2011) or Li and Matouschek (2013) for an analysis of situations where shocks to the firm are not observable to the workforce.
deviation from equilibrium-path behavior. In particular, any deviation by the principal can be punished by all agents, increasing the principal’s commitment power.

The firm’s payoff in any period $t \geq 1$ is

$$\Pi_t = E \left\{ \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left[ \pi + 1_{\{\theta_\tau > 0\}} \left( \theta_\tau f \left( \int_0^{n_\tau} e_{i_\tau} di, k \right) - \int_0^{n_\tau} \left( w_{i_\tau} + b_{i_\tau}(e_{i_\tau}, \theta_\tau) \right) di - \pi \right) \right] \right\}$$

where the indicator function describes whether the principal made offers (and hence operates the firm) in period $t$, and expectation is over the realizations of $\theta$. Furthermore,

$$\Pi_0 = -k + \delta \Pi_1.$$ 

For an agent $i$ who received an offer, the expected discounted payoff stream equals

$$U_{it} = E \left\{ \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left[ \bar{w} + d_{i_\tau} \left( w_{i_\tau} + b_{i_\tau}(e_{i_\tau}, \theta_\tau) - c(e_{i_\tau}) - \bar{w} \right) \right] \right\}.$$

We only consider pure strategies, and define a multilateral relational contract as a Subgame Perfect Equilibrium (SPE) where after every history strategies determine a Nash Equilibrium. More precisely, we are interested in a SPE that maximizes the firm’s expected profits at the beginning of the game, i.e., $\Pi_0$; we think this is a natural choice given that the labor market is assumed to be competitive, which allows firms to make take-it-or-leave-it offers to agents.

Furthermore, we can omit the index $i$. Since agents are identical and multilateral relational contracts can – and should – be used, all employees will be treated the same in equilibrium. In addition, we focus on equilibria where employment offers are accepted by employees ($d_{it} = 1$ for all $i \in [0, n_t]$), and where all employed agents choose $e = 1$ (i.e., $\int_0^{n_t} e_{i_\tau} di = n_t$). This clearly is optimal, since making an offer which is rejected and employing an agent who does not exert effort is dominated by not making an offer to any of such agents.

In Appendix A, in Lemma A1, we show that we can focus on stationary contracts that are independent of calendar time, as well as past realizations of demand shocks. Hence, employment as well as $w_t$ and $b_t$ are constant over time, allowing us to omit time subscripts $t$. This is driven by shocks being distributed i.i.d., by employment being chosen before the state of the world is realized, and by the principal – the party facing the liquidity constraint – being able to reap the whole surplus. Hence, only equilibrium bonus payments might vary over time depending on the respective realization of $\theta$. There, $b^h$ is the equilibrium bonus given $\theta^h$ is observed, and $b^l$ the bonus for $\theta^l$. Finally, we set the outside options $\pi = \bar{w} = 0.$
This assumption does not affect our results qualitatively, but simplifies our analysis given the firm’s liquidity constraint.

3 Maximization Problem and Constraints

Our objective is to find levels of capital $k$, (stationary) employment $n$, as well as a compensation package $(w, b^l, b^h)$ to maximize

$$\Pi_0 = -k + \delta \Pi,$$

where

$$\Pi = \frac{p(\theta^h f(n, k) - nb^h) + (1 - p)(\theta^l f(n, k) - nb^l) - nw}{1 - \delta}$$

is the firm’s expected discounted equilibrium payoff stream in any period $t \geq 1$.

In addition, the following constraints have to be satisfied to enforce a stationary SPE. First, it must be optimal for an agent who receives an offer to accept it. This is captured by an individual rationality (IR) constraint,

$$U \geq 0, \quad (IR)$$

where $U = w + pb^h + (1 - p)b^l - c + \delta U$ is an employed agent’s expected discounted equilibrium payoff stream and where we assume that an agent who rejects an offer does not receive another one in any subsequent period; this latter assumption will turn out to be without loss of generality.

Since employed agents are supposed to exert high effort, an incentive compatibility (IC) constraint must hold for each agent. Given promised bonus payments $b^l$ (after $\theta = \theta^l$) and $b^h$ (after $\theta = \theta^h$) are made, we have

$$pb^h + (1 - p)b^l - c + \delta U \geq 0. \quad (IC)$$

We assume that an agent is fired (and never re-hired) after selecting $e = 0$. Note that given $w \geq 0$, (IR) is automatically implied by the (IC) constraint, and we can omit the former in the following.

It must be in the interest of the principal to actually pay out $b^l$ and $b^h$ to each agent,

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5 This is based on the assumption that once an agent deviated, the principal assumes he will not exert effort in the future as well. The analysis would be identical, though, if the principal believed that an agent’s deviation was a singular event, which is driven by agents not receiving a rent in any profit-maximizing equilibrium (derived below).
which is characterized by dynamic enforcement (DE) constraints. There, if she fails to make a promised payment (which never happens in equilibrium), we assume a reversion to the static Nash equilibrium.\footnote{Following Abreu (1988): The static Nash Equilibrium determines the lower bound on the principal’s profits and should hence constitute her punishment following observable deviations.} This is characterized by no bonuses paid to any agent, who in return all choose \( e = 0 \) given they are hired. The principal then is indifferent between shutting down or running the firm without profits, in any case her off-equilibrium payoff stream is zero. Informally speaking, the firm’s failure to keep promises made to one agent leads to a loss of trust of the whole (current and prospective new) workforce.\footnote{Note that even if prospective agents were not able to observe the principal’s previous actions, merely receiving an offer could serve as a signal of an earlier deviation of the principal.} Therefore, the firm either rewards all agents or none.

The two dynamic enforcement (DE) constraints, one for \( b^l \) and one for \( b^h \), are

\[
-nb^l + \delta \Pi \geq 0 \quad \text{(DEl)}
\]

and

\[
-nb^h + \delta \Pi \geq 0. \quad \text{(DEh)}
\]

In addition, running the firm (and making employment offers) has to be optimal for the firm in every period, i.e. \( \Pi \geq 0 \). Because of stationarity and given that bonus payments are non-negative, this condition is automatically implied by the firm’s (DE) constraints. Since the right hand sides of (DEl) and (DEh) are identical, only one of them has to be considered, depending on whether \( b^l \) or \( b^h \) is larger.

Finally, payments must not violate the firm’s liquidity (L) constraints, which state that payments in any period cannot exceed respective revenues:

\[
n (w + b^l) \leq \theta^l f(n, k) \quad \text{(Ll)}
\]

and

\[
n (w + b^h) \leq \theta^h f(n, k). \quad \text{(Lh)}
\]

\section{Equilibrium Employment \( n^* \)}

In this section, we derive (profit-maximizing) equilibrium employment – denoted \( n^* \) – and how it is affected by the firm’s liquidity constraints. We show that the latter induce risk-
aversion-like behavior: For a given level of expected profits, the firm would always (weakly) prefer a lower volatility.

As a first step, we can show that (IC) constraints bind in any profit-maximizing equilibrium, and that it is further optimal to set \( w = 0 \) (see Lemma A1 in Appendix A). These results follow from the observability of effort and the excess supply of agents in the labor market, allowing the firm to reap the entire surplus from its employment relationship in any profit-maximizing equilibrium. Hence, the firm will try to maximize the total surplus subject to the constraints derived above.

### 4.1 Benchmark Cases

#### 4.1.1 Benchmark I – First-Best Employment

As a benchmark we derive the value of \( n \) that maximizes the total surplus. In the following, we denote this efficient – or first-best – employment level \( n^{FB} \). For a given capacity level \( k \), \( n^{FB} \) is characterized by

\[
(p\theta^h + (1 - p)\theta^l)f_1(\cdot, \cdot) - c = 0. \tag{FB}
\]

To keep the analysis interesting, we impose Assumption 1, implying that running the firm is strictly preferred to not running it. Given the concavity of the production function, this is also the case if first-best effort is not enforceable.

**Assumption 1:** \( f(n^{FB}, k) \left( p\theta^h + (1 - p)\theta^l \right) - n^{FB}c > 0 \) for all levels of \( k \geq 0 \). Furthermore, there exists a \( \tilde{k} > 0 \) such that \( \delta \frac{f(n^{FB}, \tilde{k}) \left( p\theta^h + (1 - p)\theta^l \right) - n^{FB}c}{1 - \delta} - \tilde{k} > 0 \)

Furthermore, we want the firm’s liquidity constraints to potentially have some bite, hence we impose

**Assumption 2:** \( n^{FB}c > \theta^l f(n^{FB}, k) \)

If Assumption 2 was violated, the firm would never be restricted by constrained liquidity and could always set \( b^h = b^l = c \).

#### 4.1.2 Benchmark II – Verifiable Effort

As another benchmark, we consider the situation where effort is verifiable but the liquidity constraints are in force. In this case, employment will in any case be at its efficient level. Hence, the firm’s constrained liquidity alone does not yield any restrictions with respect to enforceable employment, only in interaction with the non-verifiability of effort.
Lemma 1 Assume that effort is verifiable. Then, equilibrium employment equals $n^{FB}$.

Proof: See Appendix B.

The intuition is straightforward: Given that the firm has no commitment problems, it can choose to pay the agent in whatever state of the world it wants to. This allows the firm to pay mostly in those states of the world where it is not liquidity constraint (i.e. after high profits) and liquidity constraints in individual states of the world are never a problem.

4.2 Restricted Employment

If effort is not verifiable, employment can be inefficiently low, namely when one of the remaining constraint binds. This is made precise in Proposition 1.

Proposition 1 For a given capital level $k$, the firm chooses equilibrium employment $n^*$ to maximize $\Pi = \frac{(p^h + (1-p)\theta^l) f(n,k) - nc}{(1-\delta)}$, subject to

$$n^* c \leq \frac{\delta p^2}{1 - \delta} + \theta^h f(n^*, k) + (1 - p)\theta^l f(n^*, k),$$

(DE-L)

and

$$n^* c \leq \delta f(n^*, k) \left( p^h + (1 - p)\theta^l \right).$$

(DE)

There exist values $\delta = \frac{\theta^l}{(p^h + (1-p)\theta^l)}$ and $\delta$, with $\delta < \delta < 1$, such that

- $n^* = n^{FB}$ for $\delta \geq \delta$
- $n^* < n^{FB}$ for $\delta \leq \delta < \delta$, and $n^*$ is determined by the binding (DE-L) constraint
- $n^* < n^{FB}$ for $\delta < \delta$, and $n^*$ is determined by the binding (DE) constraint.

Proof: See Appendix B.

Concerning the intuition behind Proposition 1, note that a firm not facing a limited liability constraint would (optimally) set $w = 0$ and $b^h = b^l = c$ and then only be constrained by the (DE) constraint, to an employment such that $nc \leq \delta \Pi$, i.e.,

$$nc \leq \delta \left( p^h + (1 - p)\theta^l \right) f(n, k).$$

For a cash constrained firm, this is feasible (and optimal) as long as the liquidity constraints are not violated, i.e., as long as $nc \leq \theta^l f(n, k)$. Liquidity constraints are thus not binding as
long as $\delta \left( p\theta^h + (1 - p)\theta^l \right) \leq \theta^l$, i.e., for $\delta \leq \frac{\theta^l}{(p\theta^h + (1 - p)\theta^l)} = \bar{\delta}$. On the other hand, for discount factors above $\bar{\delta}$, the (DE) constraint is not binding, $b^h > b^l$, and enforceable employment restricted by the (DE-L) constraint. This case is relevant because by Assumption 2, $cn^{FB} > \theta^l f(n^{FB}, k)$, therefore $b^h > b^l$ for $n^{FB}$. The (DE-L) constraint is relaxed for larger values of $\delta$, hence there is a discount factor $\delta$ – characterized by the binding (DE-L) constraint for $n^{FB}$ – such that $n^{FB}$ can be implemented if $\delta \geq \delta$.

Put differently, if $\delta$ is between $\bar{\delta}$ and $\bar{\delta}$, the firm faces a commitment problem which is not solely created by its discounted future surplus being too low. Since constrained liquidity restricts the firm’s ability to compensate agents in low-demand states, total bonus payments are higher in the good than in the bad state. However, the principal’s willingness to reward agents is determined by the expected discounted future surplus, not by current profits. Therefore, the firm’s reneging temptation is higher in the good than in the bad state, which restricts employment given that the expected discounted future surplus is not high enough. Note that $\delta$ not only depends on the surplus of the relationship (as $\bar{\delta}$ does), but also on the volatility of earnings. This aspect will be made more precise below.

To sum up the results so far, $\hat{\delta} = \frac{\theta^l}{(p\theta^h + (1 - p)\theta^l)}$ determines whether (DE-L) or (DE) is the relevant constraint. Fixing $\theta^l$, $\hat{\delta}$ decreases in $p$ and $\theta^h$. Thus, higher values of $p$ or $\theta^h$ make it more likely that the firm is affected by (DE-L), i.e. its employment might also be restricted by a cash shortage and not only by its future returns being too low.

### 4.3 Volatility Matters

In a next step, we flesh out that it is not only players’ impatience (i.e. a low $\delta$) that limits commitment if the firm faces a liquidity constraint. This is different from “standard” contributions like MacLeod and Malcomson (1989) or Levin (2003), and also Halac (2015), where the enforceability of relational contracts is solely determined by the discount factor. We can show that the liquidity constraint will bind – and hence the firm’s employment level is restricted to inefficiently low levels – even if the principal is arbitrarily patient. This is the case if the firm’s cash-flow is very uncertain in the sense that there is a small probability that the firm makes very high profits.

**Proposition 2** Assume $\delta \geq \bar{\delta}$, i.e., the (DE) constraint (DE) can be omitted. Then, for fixed employment $\hat{n} > \frac{\theta^l f(n, k)}{c}$, $\theta^l$ and $c$, as well as for a fixed per-period surplus $f(n) \left( p\theta^h + (1 - p)\theta^l \right) - nc$, there exists a $p$ such that for $p < p$, constraint (DE-L) does not hold for $\hat{n}$.

**Proof:** See Appendix B.
A reduction of $p$, accompanied by an increase of $\theta^h$ in order to keep the surplus for a given employment level fixed will eventually lead to a violation of (DE-L).\footnote{Note that given employment, $\theta^l$, $c$ and surplus stay constant, also $\delta$ is unaffected.} This result is driven by the combination of liquidity and dynamic enforcement constraints. Absent (Ll), the enforceability of an employment level $n$ would – for a given discount factor $\delta$ – only depend on the future surplus, independent of the exact specification of $p$ and $\theta$. If a large surplus is accompanied by a larger variability of income, the principal’s commitment is still restricted. The reason is that when constrained liquidity has bite, larger shares of the compensation package must be shifted to high-demand states, ceteris paribus increasing the temptation to renege. Hence, the (DE-L) constraint is more likely to bind if the expected surplus generated in the relationship is high, and if the firm operates in a high-risk/high return environment.

In addition, we can show that given $\delta \geq \bar{\delta}$ and given the surplus is held constant, enforceable employment increases in $p$. Define $\tilde{n}$ as the employment level where constraint (DE-L) binds, i.e., $\tilde{n}$ is characterized by

$$\tilde{n}c = \frac{p^2\delta}{1-\delta + p\delta} \theta^h f(\tilde{n}) + (1-p)\theta^l f(\tilde{n}).$$

Due to the concavity of $f(n)$ in $n$, $\tilde{n}$ exists, (DE-L) is satisfied for $n \leq \tilde{n}$, and violated for $n > \tilde{n}$. Hence, for $\tilde{n} < n^{FB}$ only an inefficiently low employment level can be enforced. Building on this, we can establish Lemma 2.

**Lemma 2** Fix $\theta^l$ and the total per-period surplus (given positive effort),

$$f(n) \left( p\theta^h + (1-p)\theta^l \right) - nc.$$  

Then, $\tilde{n}$ is strictly increasing in $p$.

**Proof:** This follows from the fact that the right hand side of (1) is increasing in $p$, given the surplus is held constant (see the proof to Proposition 1). 

To sum up, this section establishes the potential enforcement problem induced by the liquidity constraint, assuming a fixed capacity level. Even if the firm is very patient, its commitment in relational contracts with its employees is limited if its earnings are volatile: Bonus payments must vary across states, whereas the maximum amount of bonuses that the principal can commit to pay in any state of the world is determined by the expected future surplus. This implies that the highest equilibrium bonus level determines whether high effort is enforceable or not. If the likelihood of the good state is too low, the necessary bonus to
imply effort becomes so high that it is not optimal for the principal to honor his promises anymore, consequently leading to inefficiently low employment.

5 Optimal Capacity and the Scope for Over-Investments

We now derive the firm’s optimal capital choice. Because capital relaxes both enforceability constraints, over-investments will occur whenever either constraint (DE) or (DE-L) binds.\(^9\)

At the beginning of the game, the firm determines its optimal capital investments \(k^*\) to maximize \(\Pi_0 = -k + \delta \Pi\), taking into account that the constraints

\[
n^*c \leq \delta f(n^*, k^*) \left(p \theta^h + (1 - p) \theta^l\right)
\]

and

\[
n^*c \leq \frac{\delta p^2}{1 - \delta + \delta p} \theta^h f(n^*, k^*) + (1 - p) \theta^l f(n^*, k^*)
\]

have to be satisfied in all later periods.

As a benchmark, the optimal capacity level absent the constraints is implicitly defined by

\[
\frac{\partial \Pi_0}{\partial k} = -1 + \frac{\delta}{1 - \delta} \left(p \theta^h + (1 - p) \theta^l\right) f_k = 0.
\]

If one of the constraints binds, though, the firm chooses capital investments that are higher than specified by (2).

**Proposition 3** Optimal capital investments are characterized by (2) if and only if efficient employment \(n^{FB}\) is enforceable, i.e., if \(\delta \geq \bar{\delta}\) (as defined in Proposition 1). If \(\delta < \bar{\delta}\), i.e., if either (DE) or (DE-L) binds, over-investments into capital are optimal in the sense that marginal costs exceed its direct marginal returns, i.e. \(k^*\) exceeds the level specified by (2).

**Proof:** See Appendix B.

The intuition underlying Proposition 3 is straightforward. If \(\delta < \bar{\delta}\), i.e., the (DE) constraint binds, ex-ante over-investments increase revenues in all future periods. Since investment costs are sunk, the difference between profits in and out-of equilibrium increases, and

\(^9\)In order to keep the analysis simple, we assume that owners of these firms are passive. They put up money to start the firm (buy the machines) but then do not interfere in running the firm (and hence are also not willing to inject money interim to relax liquidity constraints).
promises made to the firm’s workforce become more credible.\textsuperscript{10} If $\delta \geq \bar{\delta}$ (but below $\bar{\delta}$) and the (DE-L) constraint binds, over-investments are optimal because a higher value of $k$ increases output and thereby the available cash-flow in each state of the world. Then a larger share of total compensation can be shifted to the low state, thereby relaxing the firm’s (DEh) constraint.

In contrast to the classic corporate finance literature - see, e.g., Hart and Moore (1995) or Zwiebel (1996), where a reduction of free cash-flow is regarded as a potential remedy to overcome overinvestment problems, we show that the very lack of free cash-flow in some states renders over-investments ex-ante optimal.

6 Extensions and Robustness

6.1 Asset with Positive Outside Value

In this section, we relax the assumption that the initial investment is completely sunk and assume that its sale generates a positive value after a liquidation of the firm. I.e., we now assume that the investments are no longer fully relationship specific. Generally, this makes it more difficult to enforce a given employment level: If the firm reneges on promised bonus payments, its future profits are not zero anymore, but it can still enjoy the liquidation value of the asset; hence, the firm’s reneging temptation increases. With regard to over-investments, the implications are twofold: On the one hand, the benefits of over-investments are muted because a larger value of the asset also increases the firm’s reneging temptation. On the other hand, the firm’s constraints are tighter and binding earlier (i.e., for a larger value of $\delta$) if the outside value of the asset is positive – which increases the benefits of over-investments.

Formally, assume an investment $k$ has an outside value $\gamma k$, with $0 \leq \gamma \leq 1$. Hence, the firm’s dynamic enforcement constraints are $-nb^h + \delta \Pi \geq \gamma k$ and $-nb^l + \delta \Pi \geq \gamma k$.\textsuperscript{11} The firm’s limited liability constraints are unaffected. By the same considerations as above, the relevant constraints\textsuperscript{12} now are

\[ nc \leq \delta (p \theta^h + (1 - p) \theta^l) f(n, k) - (1 - \delta) \gamma k \]  

\textsuperscript{10}A similar argument is used by Halac (2015) to show that the hold-up problem (generating under-investments into relationship-specific assets) can be less severe in a relationship where relational contracts have to be used ex-post.

\textsuperscript{11}Note that we assume that upon a liquidation, the liquidation value of the asset is immediately obtained. Any delay in this process can be captured by a lower value of $\gamma$.

\textsuperscript{12}Note that we do not need an extra condition stating that a liquidation of the firm must not be optimal. Such a condition equals $\gamma k \leq \delta \Pi$ and is implied by $\Pi_0 > 0$. 

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\[ nc \leq (1 - p)\theta^l f(n, k) + p(\delta \Pi - \gamma k) \]. \hspace{1cm} (DE-L)

It is straightforward to see that both constraints are tighter for larger values of \( \gamma \). Ceteris paribus, a lower degree of asset specificity, i.e., a larger value of \( \gamma \), implies that lower employment levels can be sustained.

Optimal employment and investment levels, as well as the effect of the degree of asset specificity \( \gamma \) on (over-) investments are given in Lemma 4:

**Proposition 4** There exist values \( \delta(\gamma) \) and \( \overline{\delta}(\gamma) \), with \( \delta(\gamma) < \overline{\delta}(\gamma) < 1 \), such that

- \( n^* = n^{FB} \) for \( \delta \geq \delta(\gamma) \); in this case, capacity investments are characterized by (2)
- \( n^* < n^{FB} \) for \( \delta < \delta(\gamma) \), and \( n^* \) is determined by the binding (DE-L) constraint; in this case, over-investments are optimal.
- \( n^* < n^{FB} \) for \( \delta < \overline{\delta} \), and \( n^* \) is determined by the binding (DE) constraint; in this case, over-investments are optimal for \( \gamma < 1 \).

Furthermore, \( d\delta(\gamma)/d\gamma > 0 \) and \( d\overline{\delta}(\gamma)/d\gamma > 0 \); if either (DE) or (DE-L) binds, investments might increase or decrease in \( \gamma \).

**Proof:** See Appendix B.

The equilibrium outcome described here resembles the one with \( \gamma = 0 \): For rather low discount factors, (DE) is the relevant condition and employment is only constrained by future profits being too small. For intermediate discount factors, (DE-L) is the relevant condition and employment is also constrained by the firm having insufficient liquidity in bad states.

Over-investments remain optimal for \( \delta < \delta(\gamma) \), unless \( \gamma = 1 \) and (DE) is binding \((\delta < \delta(\gamma))\). In this case, \( \gamma = 1 \) implies that the asset is not relationship specific at all, and one dimension of the benefits of over-investments vanishes: Those relax the (DE) constraint in our baseline settings because they increase the difference between the firm’s future on- and off-path rents and thereby reduce the firm’s temptation to renege on bonus payments. However, this mechanism loses its validity if \( \gamma = 1 \) because the firm’s off-path rents contain a liquidation of the asset which is not costly in this case. If the (DE-L) constraint binds, on the contrary, over-investments remain optimal even if the asset is not relationship-specific – because higher investments provide additional liquidity on the equilibrium path.

Concerning the marginal effect of the asset’s relationship specificity on optimal investments, note the following: First, critical discount factors are increasing in \( \gamma \), hence a larger
\( \gamma \) increases the parameter range where constraints bind and over-investments are generally optimal. This is because a larger \( \gamma \) increases the firm’s outside option and hence tightens the relevant constraints. If \( \delta < \overline{\delta}(\gamma) \), a higher \( \gamma \) has two opposing effects. On the one hand – as just mentioned – it tightens the firm’s constraints, which amplifies the benefits of over-investments. On the other hand, the increase in the firm’s outside option is more pronounced for a larger \( k^* \), hence a larger \( \gamma \) reduces the benefits of over-investments. The latter effect is more pronounced in case the (DE) constraint binds, i.e., for rather low discount factors. For intermediate discount factors such that (DE-L) is the relevant constraint, the marginal effect of \( \gamma \) on \( k^* \) is more likely to be positive.

\section*{6.2 Firms Have Access to a Credit Market}

In this section, we explore whether the availability of a credit market can solve the problem of limited liability. We argue that even if a competitive credit market where repayment can be contingent on the state of the world exists, over-investments remain optimal for many firms. Potentially, a firm can benefit from a credit market if its (DE-L) constraint binds (if the DE constraint binds, i.e., if employment is solely restricted by the firm’s future being too low, borrowing obviously does not help). Then, the limited liability constraint could be relaxed by borrowing in low-demand and repaying loans in high-demand states.

First, we show that the existence of a credit market has no effect in our baseline model where the firm’s asset has no outside value – and cannot be used as collateral. Only if this outside value is positive (i.e., if \( \gamma > 0 \) as introduced in the previous section), the credit market can be used to partially relax the firm’s limited liability constraint. In the following, we will assume an intermediate discount factor, i.e., the (DE-L) constraint binds (given no credit is taken by the firm) whereas (DE) is slack.

\subsection*{Asset has no Outside Value}

Generally, the firm needs an incentive to repay its debt – assuming that a default leads to a termination of the firm or that at least the principal has no access to any future profits generated by it. In the following, assume there is a competitive credit market for short-term credit where creditors also have a discount factor \( \delta \), and that repayment can be made contingent on the state of the world. Furthermore, repayment is required in every high-demand period (even if the firm has not borrowed in the previous period; this “smoothing” of repayments is optimal because of the firm’s DE constraint in the high-demand state).

We denote the amount borrowed by the firm in a low-demand state by \( D \), and the demand repaid in a high-demand state by \( R \). Because the credit market is competitive, \( pR - (1-p)D = \)
0, hence \( R = (1 - p)D/p \).

First, note that for a given employment level, the principal profits are naturally unaffected by the existence of the credit market,
\[
\Pi = \frac{p(\theta^f(n,k) - nb^h - R) + (1 - p)(\theta^f(n,k) - nb^l + D)}{1 - \delta},
\]
where by construction \( pR = (1 - p)D \).

However, the credit market could increase profits if it allowed to increase feasible employment. To assess this aspect, recall that the constraints that determine \((DE-L)\) are \((Ll)\) and \((DEh)\), which – taking the consequences of a credit market into account – now amount to
\[
nb^l \leq \theta^l f(n,k) + D \quad (Ll)
\]
\[
-nb^h - R + \delta \Pi \geq 0 \quad (DEh)
\]

The firm also needs an incentive to make the payment \( R \) (i.e., \( -R + \delta \Pi \) must hold), which however is implied by \((DEh)\). Enforceable employment again is determined by the \((DE-L)\) constraint, which is obtained by adding \((Ll)\) (multiplied with \( 1 - p \)) and \((DEh)\) (multiplied with \( p \)), taking into account that an agent’s \((IC)\) constraint will bind. Then, however, \( D \) and \( R \) cancel out and \((DE-L)\) looks exactly as before – and a credit market does not expand enforceable employment.

To understand the intuition behind this result, recall that without a credit market, a binding limited liability constraint implies that bonus payments are larger in high- than in low-demand states (thereby tightening the \(DE\) constraint in high states). Having a credit market allows to smooth bonuses, however the effective total payments made by the firm are still equivalently large in high-demand states – because creditors demand their repayments. Furthermore, the firm’s incentives regarding paying the bonus and compensating creditors are identical – refusing to make either payment triggers zero continuation profits.

**Asset has Positive Outside Value**

When the asset has a positive outside value, it can serve as collateral and the credit market can effectively be used to smooth payments to agents and thereby relax the firm’s limited liability constraint. Then, a credit market can actually help to enforce higher employment levels, however over-investments will still be optimal for many firms. In the following, we assume a credit market as laid out above (a credit market with an interest larger than the rate reflecting time preferences, or with repayment not being fully contingent on the state of the world reduces its benefits and gets us closer to our baseline situation).

As long as repayment obligations \( R \) do not exceed the liquidation value of the asset, \( \gamma k \),
credit now does not tighten the firm’s (DEh) constraint. This is because the firm cannot avoid paying $R$ by defaulting, hence the (DEh) constraint becomes $-nb^h - R + \delta \Pi \geq \gamma k - R$. The firm’s bad-state limited liability constraint is relaxed because more cash is available. Consequently, the (DE-L) constraint is relaxed and becomes

$$-nc + \delta p \Pi + (1 - p) \theta^l f(n, k) + (1 - p) D - p \gamma k \geq 0.$$ 

Therefore, the left hand side of the (DE-L) constraint is maximized for choosing $D = \frac{p}{(1-p)} \gamma k$, i.e., such that $R = \gamma k$ (larger values cannot be collateralized and hence do not further relax the constraint).

This implies that the firm uses the credit market if the (DE-L) constraint binds in order to increase its employment level. It will either borrow an amount smaller than $\frac{p}{(1-p)} \gamma k$, such that the (DE-L) constraint just does not bind anymore (and either $n^{FB}$ can be implemented or (DE) has become the relevant constraint), or it will borrow an amount $D = \frac{p}{(1-p)} \gamma k$ (if the (DE-L) still binds for this debt level).

Overinvestments are still optimal unless $n^{FB}$ can be implemented. In case the (DE-L) constraint binds for $D = \frac{p}{(1-p)} \gamma k$, there actually is an additional benefit of over-investments: those relax the constraint $D \leq \frac{p}{(1-p)} \gamma k$, hence having a larger value of $k$ creates more collateral and allows to take more debt in order to smooth payments.

### 6.3 Firms can Accumulate Retained Earnings

In this section, we show that over-investments can still be optimal if the firm is able to retain its earnings in high-demand periods in order to relax the liquidity constraint in later periods (within our baseline model where the asset has no outside value and no credit market exists). Doing so allows the firm to temporarily increase its employment level, and might or might not be optimal if the firm’s liquidity constraint binds, but can never make it irrelevant. Hence, over-investments continue to be part of the firm’s optimal investment decisions whenever employment is restricted.

To simplify the analysis, we assume that accumulated cash stays within the firm and does not generate interest payments. If the firm decides to retain earnings, the accumulated cash can be used in two ways: On the one hand, employment can be increased until the next realization of $\theta^l$. On the other hand, a given employment level can be sustained longer, i.e. for more than just one subsequent realization of $\theta^l$. We relegate a general characterization of the firm’s optimal behavior to Appendix C and only present the main result in this section.
Proposition 5  Retaining earnings is optimal if and only if in the situation without this possibility, the (DE-L) constraint binds (i.e., if $\delta$ is between $\bar{\delta}$ and $\tilde{\delta}$, as defined above in Proposition 1) and the following condition is satisfied:

$$
\left( \frac{\delta (1 - p)}{1 - \delta p} p^{\theta^h} + (1 - p) \theta^l \right) f_n - c > 0.
$$

(3)

In this case, maximum employment is characterized by

$$
\left( \frac{\delta (1 - p)}{1 - \delta p} p^{\theta^h} + (1 - p) \theta^l \right) f_n - c = 0.
$$

(4)

Proof: See Appendix C.

Holding cash reserves is never optimal if only (DE) constraints bind. However, a binding liquidity constraint alone is only a necessary and not a sufficient condition for savings to be optimal. Instead, the costs of holding cash reserves – delayed consumption – might still be too high compared to the benefits of having higher employment in the future, which is the case if condition (3) does not hold.

In Appendix C, we further show that after a cash stock has been built up and the firm is hit by a number of negative shocks, employment is decreased gradually. Then, each low-demand period triggers an employment reduction until all savings have been used up. This parallels results in Li and Matouschek (2013) where implemented effort levels gradually decrease with every adverse shock hitting a firm.

Finally, we can show that the possibility to accumulate cash reserves does not eliminate over-investments.

Proposition 6  Assume retaining earnings is possible. Then over-investments are optimal if in the situation without this possibility, either (DE) or (DE-L) constraints bind.

Proof of Proposition 6: If (DE) constraints bind, holding cash is not optimal and the situation is equivalent to above – over-investments are an optimal response to relax the constraint. If holding is optimal due to a binding (DE-L) constraint and because condition 3 holds, employment never exceeds the level characterized by condition (4) – hence is below the first best. The rest follows directly from the proof to Proposition 3.
7 Discussion and Conclusion

The present paper shows that observed over-investments are not necessarily the (negative) consequence of agency problems between shareholders and managers. Instead, they might actually be a second-best-optimal response to contracting frictions: In situations where firms face volatile market conditions and hence varying cash-flow streams, and where they cannot rely on court-enforceable contracts to motivate their workforce but have to use relational contracts instead, “excessive” capital investments relax liquidity constraints by increasing the firm’s cash-flow base.

In the following we relate our results to extant findings in the literature and present avenues for future research.

The literature has documented how financial frictions restrict firms’ choices. Because external financing is generally regarded as more expensive than internal financing, the unavailability of sufficient internal funds – for example caused by a high cash flow volatility – can be a serious impediment to firm growth and reduce future profits (cf. Huang (2009)). If cash flow volatility is high, firms face times of low (internally generated) liquidity. It has been documented widely that these cash flow shortages trigger a reduction of firms’ future investment; see Fazzari, Hubbard, Petersen, Blinder, and Poterba (1988); Hoshi, Kashyap, and Scharfstein (1991); Kaplan and Zingales (1997); and Minton and Schrand (1999). We argue that periods of low cash flow might not only restrain firms’ growth opportunities, but also make it more difficult to employ the efficient amount of labor.

Internal capital markets have been identified as an instrument to mitigate a firm’s exposure to (inefficient) external capital markets. If firms pool their resources, they might use their funds more efficiently and thereby improve capital allocation; see Stein (1997), or Inderst and Laux (2005). In our setting merging two firms whose demand conditions are not perfectly correlated would allow them to relax the pooled limited liability constraint in instances where one firm’s demand condition is low and the other’s is high. Therefore, internal capital markets can have an additional, so far not highlighted role: they can also serve as an instrument for firms to manage their workforce more efficiently.

Empirically, Bargeron, Schlingemann, Stulz, and Zutter (2008) show that publicly listed firms in the US pay substantially more for acquisitions than privately owned firms and argue that this might be due to larger agency problems in publicly-traded firms. Our model can provide a complementary explanation: Privately held firms are often assumed to focus more on long-term goals – in particular with respect to their employment relationships – compared to publicly listed firms. Take family firms, where for example a study by Price Waterhouse Cooper (2012) identifies a larger commitment to jobs, which leads “family-run businesses ... to
have more loyalty toward their staff – people are not just a number” (PriceWaterhouseCooper (2012), p. 6). In terms of our model, privately-held firms hence would have a larger discount factor and benefit less from over-investments than publicly traded firms (over-investments are optimal given \( \delta < \bar{\delta} \), and their amount decreases in \( \delta \)).\(^{13}\) This would imply that publicly traded firms – ceteris paribus – have a higher willingness to pay for a given acquisition target, as documented by Bargeron, Schlingemann, Stulz, and Zutter (2008).

So far, we did not allow for different kinds of investments. In our setting, firms facing binding (DE-L) constraints, prefer investment opportunities with less volatile cash flows, even at the cost of lower expected returns. I.e., these firms would abstain from making R&D-type (high-risk/high return) investments. Hence, there is potentially an additional indirect cost of quality of low contract enforcement in a country: reduced R&D activity and on the macro level reduced growth.\(^{14}\)

The predictions of our model can be matched to empirically documented regularities. Proposition 1 also implies that on average, firms with a higher discount factor have a larger variation in residual profits and hence free cash flow: Given \( \delta \) is between \( \bar{\delta} \) and \( \bar{\delta} \), then \( b'n = \theta^l f(n, k) \), and the firm has to use all of its revenues to compensate agents if demand is low. On the other hand, if \( \delta < \bar{\delta} \), the liquidity constraint does not bind, and free cash flow is also available in the low-demand state. If we assume that dividend payments (or more generally the distribution of profits to owners/shareholders) are larger in states when more cash-flow is available, we should expect firms with higher discount factor to have a larger variability in dividend payments. This matches the finding in Michaely and Roberts (2012) that privately held firms in the UK smooth dividends significantly less than their publicly listed counterparts, and respond more to transitory earnings shocks. Michaely and Roberts (2012) conjecture that this might be due to agency problems, which are more prominent in publicly held firms. We offer an alternative but related explanation, again because of the aforementioned notion that privately-held firms are supposed to have a larger discount factor. This makes it more likely that their dynamic enforcement constraint (DE-L) is the relevant constraint, implying more variation in dividend payments.

Finally, Lemma 2 has a testable prediction: If \( \theta^h \) is very high, but the prospects of it being realized are low, enforceable employment is smaller than with a lower \( \theta^h \) but higher probability \( p \). We should thus observe that industries which ceteris paribus are more volatile have lower employment levels than industries where earnings are more balanced. We are not aware of a study that has looked into this, but the prediction appears readily testable.

\(^{13}\)Note that this result would still hold absent a liquidity constraint.

\(^{14}\)We are grateful to Bob Gibbons for pointing out this implication.
A Maximization Problem, Constraints, and Proof of Stationarity

Note that it is sufficient to regard equilibrium employment as well as compensation as a function of the history of past shocks. The reason is our focus on pure strategies. Denote the history at the beginning of period $t$ as $\theta^{t-1} = \{\theta_1, \theta_2, ..., \theta_{t-1}\}$, with $\theta_t \in \{\theta^l, \theta^h\}$, and $\theta^0 = \emptyset$. Then, expected payoff streams can be written as

$$\Pi(\theta^{t-1}) = p \left[ \theta^h f (n(\theta^{t-1}), k) - n(\theta^{t-1})b^h(\theta^{t-1}) \right]$$

$$+ (1 - p) \left[ \theta^l f (n(\theta^{t-1}), k) - n(\theta^{t-1})b^l(\theta^{t-1}) \right]$$

$$- n(\theta^{t-1})w(\theta^{t-1}) + \delta \left[ p\Pi(\theta^{t-1}, \theta^h) + (1 - p)\Pi(\theta^{t-1}, \theta^l) \right] +$$

and

$$U(\theta^{t-1}) = w(\theta^{t-1}) + pb^h(\theta^{t-1}) + (1 - p)b^l(\theta^{t-1}) - c$$

$$+ \delta \left[ pU(\theta^{t-1}, \theta^h) + (1 - p)U(\theta^{t-1}, \theta^l) \right];$$

where $b^h(\theta^{t-1})$ is the bonus paid for history $\theta^{t-1}$ given a high shock is realized in period $t$. Equivalent definitions hold for $b^l(\theta^{t-1})$, $\Pi(\theta^{t-1}, \theta_t)$ and $U(\theta^{t-1}, \theta_t)$.

Then, the firm’s objective function is to choose $k$ as well as $n(\theta^{t-1})$, $w(\theta^{t-1})$, $b^h(\theta^{t-1})$ and $b^l(\theta^{t-1})$ to maximize

$$\Pi_0 = -k + \delta\Pi(\theta^0),$$

subject to the following constraints, which must be satisfied for every history $\theta^{t-1}$:

$$U(\theta^{t-1}) \geq 0$$

$(\text{IRA})$

$$pb^h(\theta^{t-1}) + (1 - p)b^l(\theta^{t-1}) - c + \delta \left[ pU(\theta^{t-1}, \theta^h) + (1 - p)U(\theta^{t-1}, \theta^l) \right] \geq 0$$

$(\text{IC})$

$$n(\theta^{t-1})b^l(\theta^{t-1}) \leq \delta\Pi(\theta^{t-1}, \theta^l)$$

$(\text{DEl})$

$$n(\theta^{t-1})b^h(\theta^{t-1}) \leq \delta\Pi(\theta^{t-1}, \theta^h)$$

$(\text{DEh})$

Given both $(\text{DE})$ constraints, the firm’s individual rationality constraint, $\Pi(\theta^{t-1}) \geq 0$, is automatically satisfied.
\[ n(\theta^{t-1}) (w(\theta^{t-1}) + b'(\theta^{t-1})) \leq \theta^t f(n(\theta^{t-1})) \]  
\begin{equation}
\text{(Ll)}
\end{equation}

\[ n(\theta^{t-1}) (w(\theta^{t-1}) + b^h(\theta^{t-1})) \leq \theta^h f(n(\theta^{t-1})) \]  
\begin{equation}
\text{(Lh)}
\end{equation}

**Lemma 3** (IC) constraints bind, and \( w(\theta^{t-1}) = 0 \) for every history \( \theta^{t-1} \).

**Proof of Lemma 3:** To the contrary, assume there is a history \( \tilde{\theta}^{t-1} \) where (IC) does not bind. At this point, reduce \( b^h(\tilde{\theta}^{t-1}) \) as well as \( b'(\tilde{\theta}^{t-1}) \) by a small \( \varepsilon > 0 \) such that (IC) for history \( \tilde{\theta}^{t-1} \) is still satisfied. Furthermore, increase \( w(\tilde{\theta}^{t-1}) \) by \( \varepsilon \) and leave everything else unchanged. This has no impact on \( \Pi_0 \), as well as \( \Pi(\theta^{t-1}) \) and \( U(\theta^{t-1}) \) for any history \( \theta^{t-1} \), hence does not affect any (IRA) constraint. Furthermore, all (Ll) and (Lh) constraints remain unchanged. Finally (DEh) and (DEl) for history \( \tilde{\theta}^{t-1} \) are relaxed and unaffected for any other history.

Plugging this result into \( U(\theta^{t-1}) \) gives \( U(\theta^{t-1}) = w(\theta^{t-1}) \); hence, setting \( w(\theta^{t-1}) = 0 \) for all histories \( \theta^{t-1} \) is optimal, since it relaxes the firm’s constraints and increases \( \Pi_0 \).  

Using the results of Lemma A1 gives
\[ \Pi(\theta^{t-1}) = p\theta^h f \left( n(\theta^{t-1}), k \right) + (1 - p)\theta f \left( n(\theta^{t-1}), k \right) - n(\theta^{t-1})c \]
\[ + \delta \left[ p\Pi \left( \theta^{t-1}, \theta^h \right) + (1 - p)\Pi \left( \theta^{t-1}, \theta^l \right) \right] \]
and \( b^h(\theta^{t-1}) = \frac{c(1 - p)b(\theta^{t-1})}{p} \).

Furthermore, the remaining constraints are
\[ n(\theta^{t-1})b'(\theta^{t-1}) \leq \delta \Pi \left( \theta^{t-1}, \theta^l \right) \]  
\begin{equation}
\text{(DEl)}
\end{equation}

\[ n(\theta^{t-1}) \frac{c - (1 - p)b'(\theta^{t-1})}{p} \leq \delta \Pi \left( \theta^{t-1}, \theta^h \right) \]  
\begin{equation}
\text{(DEh)}
\end{equation}

\[ n(\theta^{t-1})b'(\theta^{t-1}) \leq \theta^l f \left( n(\theta^{t-1}), k \right) \]  
\begin{equation}
\text{(Ll)}
\end{equation}

\[ n(\theta^{t-1}) \frac{c - (1 - p)b'(\theta^{t-1})}{p} \leq \theta^h f \left( n(\theta^{t-1}), k \right) \]  
\begin{equation}
\text{(Lh)}
\end{equation}

This allows us to prove

**Lemma 4** Contracts are stationary in a sense that employment as well as bonus payments are independent of the history of shocks \( \theta^{t-1} \).
Proof of Lemma 4: First of all, note that none of the employment levels can optimally be above \( n_{\text{FB}} \) and that profits are increasing in \( n(\theta_{t-1}) \) for any possible history as long as employment there is inefficiently low. Now, take any equilibrium employment level \( n(\theta_{t-2}) \) and assume that \( n(\theta_{t-2}, \theta^h) \neq n(\theta_{t-2}, \theta^l) \). If \( n(\theta_{t-2}, \theta^h) > n(\theta_{t-2}, \theta^l) \), replacing \( n(\theta_{t-2}, \theta^l) \) with \( n(\theta_{t-2}, \theta^h) \) would violate no constraint and increase profits. If \( n(\theta_{t-2}, \theta^l) > n(\theta_{t-2}, \theta^h) \), on the other hand, replacing \( n(\theta_{t-2}, \theta^h) \) with \( n(\theta_{t-2}, \theta^l) \) would violate no constraint and increase profits. Hence employment in any period is independent of previous shock realizations and might only be history-dependent based on the number of observed shocks, i.e. on timing. There, however, note that the structure of the game is stationary. This implies that the highest employment level that is enforceable in any period of the game can be enforced in all other periods as well. Since the principal is able to collect the whole surplus of the game, it is optimal to choose the maximum feasible employment (subject to \( n \leq n_{\text{FB}} \)) in every period. Finally, note that we have shown in Lemma 3 that it is optimal to set \( U(\theta^l) = w(\theta^l) = 0 \) for all histories \( \theta^l \). Therefore, the binding (IC) constraint implies that it is (weakly) optimal to have \( b^h(\theta^l) \) and \( b^l(\theta^l) \) independent of the history \( \theta^l \).

Note that Lemmas 3 and 4 establish that it is not optimal to promise the workforce a higher continuation payoff in a low-demand state, i.e., paying them a rent in a future period when demand is high. The reason is that this tightens the high-state (DE) constraints equivalently and hence does not allow for higher employment levels.

B Omitted Proofs

Proof of Lemma 1: If effort is verifiable, the firm’s dynamic enforcement constraints can be omitted. We just have to make sure that the firm’s expected profits are positive.

Assume the following contract: Employment equals \( n_{\text{FB}} \), each agent who chooses positive effort receives expected bonus payments \( pb^h + (1-p)b^l = c \) and gets fired otherwise. If \( n_{\text{FB}}c \leq \theta^l f(n_{\text{FB}}, k) \), then \( b^l = b^h = c \), satisfying all constraints and obviously being optimal for the principal due to Assumption 1.

Otherwise, i.e., if \( n_{\text{FB}}c > \theta^l f(n_{\text{FB}}, k) \), the firm sets \( b^l = \theta^l f(n_{\text{FB}}, k)/n_{\text{FB}} \), and \( b^h = (c - (1-p)\theta^l f(n_{\text{FB}}, k)/n_{\text{FB}}) / p \). Plugging this value into (Lh) gives

\[
0 \leq p\theta^h f(n_{\text{FB}}, k) + (1-p)\theta^l f(n_{\text{FB}}, k) - n_{\text{FB}}c,
\]

which is always satisfied due to Assumption 1.

Proof of Proposition 1. Recall that the firm’s objective is to maximize \( \Pi_0 = -k + \delta \Pi \), subject to the relevant constraints. For a given value of \( k \), however, \( n \) is chosen to maximize
Π. Furthermore, we show in Appendix A that (IC) and (IRA) constraints bind. Hence, we can use \( w = 0 \) and \( pb^h + (1 - p)b^l - c = 0 \), giving the problem

\[
\max_{n} \Pi = \frac{p\theta^h f(n, k) + (1 - p)\theta^l f(n, k) - nc}{1 - \delta},
\]

subject to

\[
nb^l \leq \delta \Pi \quad \text{(DEl)}
\]

\[
n\frac{c - (1 - p)b^l}{p} \leq \delta \Pi \quad \text{(DEh)}
\]

\[
nb^l \leq \theta^l f(n, k) \quad \text{(Ll)}
\]

\[
n\frac{c - (1 - p)b^l}{p} \leq \theta^h f(n, k) \quad \text{(Lh)}
\]

It follows that (DEl) and (Ll), as well as (DEh) and (Lh) generally do not bind at the same time. In a next step, we show that (Lh) cannot bind in equilibrium. To the contrary assume it binds. Then, either (Ll) or (DEl) must bind as well because otherwise, \( b^l \) could be increased and (Lh) relaxed without violating any constraint. First, assume that (Ll) binds together with (Lh). This, however, would imply that \( \Pi = 0 \), which is not possible in an equilibrium with positive employment. Now, assume that (DEl) binds together with (Lh). From section 3, we know that setting \( b^h \geq b^l \) is optimal. Hence, (DEh) has to bind as well, implying \( b^h = b^l \). Then, \( \theta^h f(n, k) = nb^h = nb^l = \theta^l f(n, k) \), which - due to \( \theta^h > \theta^l \) - is not possible for \( n > 0 \).

Now, consider all employment levels with \( \theta^l f(n, k) \geq \delta \Pi \). In this case, (Ll) is automatically satisfied given (DEl). Adding (DEh) (multiplied with \( p \)) and (DEl) (multiplied with \( (1 - p) \)) proves the necessity of (DE-L). Sufficiency immediately follows: Assuming (DE-L) holds, there always exists a \( b^l \geq 0 \) such that (DEh) and (DEl) are satisfied.

For employment levels \( \theta^l f(n, k) < \delta \Pi \) (DEl) is automatically satisfied given (Ll). In this case, necessity and sufficiency of (DE) are obtained equivalently as for (DE-L).

To prove that \( n^* \leq n^{FB} \), we set up the Lagrange function,

\[
L = \frac{p\theta^h f(n, k) + (1 - p)\theta^l f(n, k) - nc}{1 - \delta} + \lambda_{DE} \left[ \delta f(n, k) \left( p\theta^h + (1 - p)\theta^l \right) - nc \right]
+ \lambda_{DEL} \left[ \frac{\delta p^2}{1 - \delta + \delta p} \theta^h f(n, k) + (1 - p)\theta^l f(n, k) - nc \right],
\]

giving the first order condition

\[
\frac{\partial L}{\partial n} = \left( p\theta^h f_n + (1 - p)\theta^l f_n - c \right) \left( \frac{1}{1 - \delta} + \delta \lambda_{DE} + \lambda_{DEL} \right)
\]
\[-\lambda_{DEc}(1 - \delta) - \lambda_{DELP}\theta^h f_n \frac{1 - \delta}{1 - \delta + \delta p} = 0.\]

Hence, if either (DE-L) or (DE) binds \((p\theta^h f_n + (1 - p)\theta^d f_n - c) > 0\), and \(n^*\) is inefficiently small.

To determine \(\delta\) and \(\delta^*\), fix employment \(n\). Furthermore, (DE-L) is the relevant constraint when the right hand side of (DE) is smaller than (DE-L) for a given employment level, or if

\[
\delta (p\theta^h + (1 - p)\theta^d) \geq \theta^d
\]

Condition (5) holding as an equility gives the value \(\delta = \theta^d / (p\theta^h + (1 - p)\theta^d)\). To show that \(n^{FB}\) cannot be enforced at \(\delta\), plug \(\delta\) into each of the constraints. This gives \(nc \leq \theta^d f(n, k)\) which – by Assumption 2 – does not hold for \(n^{FB}\).

To prove the rest of the proposition, and in particular the existence of \(\delta^*\), we need monotonicity of the constraints in \(\delta\), and that first-best employment can be enforced for \(\delta \to 1\). The latter follows from Assumption 1 ((DE-L) and (DE) both converge to \(n^*c \leq p\theta^h f(n^*, k) + (1 - p)\theta^d f(n^*, k)\) for \(\delta \to 1\)). To prove monotonicity in \(\delta\), we obtain the derivatives of the right hand sides of the (DE) and (DE-L) constraints with respect to \(\delta\), \(f(n, k) (p\theta^h + (1 - p)\theta^d)\) and \(\frac{p^2}{(1 - \delta + \delta p)^2} \theta^h f(n, k)\), respectively. Since both expression are positive, both constraints are relaxed by larger values of \(\delta\).

**Proof of Proposition 2.** As we fix the surplus, as well as \(\theta^d\) and \(n\), a decrease in \(p\) has to be compensated by an appropriate increase in \(\theta^h\). More precisely, taking the total derivative of the per-period surplus, \(f(n, k) (dp\theta^h + pd\theta^d - dp\theta^d) = 0\), implies \(\frac{dp\theta^h}{dp} = -\frac{(\theta^h - \theta^d)}{p}\).

Take an arbitrary high-state probability \(p\) where constraint (DE-L) is satisfied. For any probability \(p^* < p\), always counterbalanced by an increase of \(\theta^h\) that keeps the surplus constant, the right hand side of (DE-L) equals

\[
\frac{\delta}{1 - \delta + p^* \delta} (p^*)^2 (\theta^h + d\theta^h) f(n, k) + (1 - p^*)\theta^d f(n, k)
\]

\[
= \frac{\delta}{1 - \delta + p^* \delta} (p^*)^2 \left( \theta^h - (\theta^h - \theta^d) \int_{p^*}^p \frac{1}{p} dp \right) f(n, k) + (1 - p^*)\theta^d f(n, k)
\]

\[
= \frac{\delta}{1 - \delta + p^* \delta} (p^*)^2 (\theta^h - (\theta^h - \theta^d) \ln p + (\theta^h - \theta^d) \ln p^* + C) f(n, k) + (1 - p^*)\theta^d f(n, k)
\]

For \(p^* \to 0\), the last expression becomes

\[
\frac{\delta}{1 - \delta (1 - p^*)} (\theta^h - \theta^d) \ln p f(n, k) + \theta^d f(n, k)
\]

\[
= \frac{\delta}{1 - \delta} (\theta^h - \theta^d) \frac{\ln p f(n, k)}{(p^*)^2} + \theta^d f(n, k) = \theta^d f(n, k).
\]

Since \(\theta^d f(n, k) < nc\), the employment level \(n\) will eventually not be enforceable anymore.

Finally, we have to show that the right hand side of (DE-L) is increasing in \(p\).
\[
\frac{d}{dp} = \delta^2 p \theta^h f(n, k) + \frac{p^2 \theta}{1 - \delta + \rho} \frac{dp}{df} f(n, k) - \theta f(n, k)
\]
\[
= \frac{(1 - \delta)^2}{(1 - \delta + \rho)^2} \left[ \delta \left( \theta^h f(n, k) p + (1 - p) \theta^f f(n, k) \right) - \theta f(n, k) \right].
\]
Since \( \theta f(n, k) < cn \), this expression is positive as long as \( n \) is enforceable if the liquidity constraint is absent. However, if \( n \) is not enforceable without the principal’s liquidity constraint, \((DE-L)\) is violated anyway.

**Proof of Proposition 3.** The Lagrange function in this case equals
\[
L = -k + \frac{\delta}{1 - \delta} \left[ (p \theta^h + (1 - p) \theta^f) f(n, k) - nc \right]
+ \lambda_{DE} \left[ \delta f(n^*, k) \left( p \theta^h + (1 - p) \theta^f \right) - n^* c \right]
+ \lambda_{DEL} \left[ \frac{\delta p^2}{1 - \delta + \rho} \theta^h f(n^*, k) + (1 - p) \theta f(n^*, k) - n^* c \right],
\]
and the first order condition with respect to \( k \) is
\[
\frac{\partial L}{\partial k} = -1 + \frac{\delta}{1 - \delta} \left( p \theta^h + (1 - p) \theta^f \right) f_k + \lambda_{DE} \delta f_k \left( p \theta^h + (1 - p) \theta^f \right)
+ \lambda_{DEL} \left[ \frac{\delta p^2}{1 - \delta + \rho} \theta f_k + (1 - p) \theta f_k \right] = 0.
\]

Hence, if either \((DE)\) or \((DE-L)\) binds, \( k^* \) will satisfy \(-1 + \frac{\delta}{1 - \delta} \left( p \theta^h + (1 - p) \theta^f \right) f_k < 0\), implying a \( k^* \) that is larger than the one characterized by \((2)\).

**Proof of Proposition 4.** The relevant constraints are
\[
nc \leq \delta \left( p \theta^h + (1 - p) \theta^f \right) f(n, k) - \delta \gamma k \quad (DE)
\]
\[
n \leq \left[ (1 - p) \theta^f + \frac{\delta p^2}{1 - \delta + \rho} \theta^h \right] f(n, k) - \frac{(1 - \delta) p}{1 - \delta + \rho} \gamma k. \quad (DE-L)
\]

First note that \((DE)\) is the tighter constraint if the right hand side of \((DE)\) is smaller than then right hand side of \((DE-L)\), which holds for \( \delta^2 (\gamma) \leq \frac{\theta^f + \gamma k}{(p \theta + (1 - p) \theta^f + \gamma k) ((n, k, \delta)}} \). In a next step, we plug \( \delta \) into the \((DE)\) constraint and get
\[
\left( p \theta^h + (1 - p) \theta^f + \frac{\gamma k}{(n, k, \delta)} \right) \left( \theta^f f(n, k) - nc \right) \geq 0.
\]
This is not satisfied for \( n^{FB} \) by Assumption 2, hence \( n^{FB} \) cannot be implemented at \( \delta (\gamma) \). Furthermore, the right hand sides of \((DE)\) and \((DE-L)\) constraints are increasing in \( \delta \), and \( n^{FB} \) satisfies the \((DE-L)\) constraint for \( \delta \to 1 \). This implies that there is a critical discount factor \( \delta (\gamma) \) above which \( n^{FB} \) can be implemented (characterized by \( (1 - \delta + \rho) n^{FB} c = \left[ (1 - p) (1 - \delta + \rho) \theta^f + \delta p^2 \theta^h \right] f(n^{FB}, k) - (\delta - \rho) \gamma k \), and that \( \delta (\gamma) > \delta (\gamma) \).

Furthermore,
\[
\frac{d\delta}{d\gamma} = \frac{k}{(n, k, \delta)} \left( p \theta^h (1 - \theta^f) + \frac{\gamma k}{(n, k, \delta)} \right) \] and

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\[ \frac{\partial L}{\partial \gamma} = \frac{(1-\delta)(1-\delta+p\delta)}{p\theta h (n^{FB},k)+p\gamma k} > 0, \] where the last step uses the binding (DE-L) constraint.

In a next step, we solve for optimal employment and investment levels. The Lagrange function of the maximization problem equals

\[
L = -k + \frac{\delta}{1-\delta} \left[ (p\theta h + (1-p)\theta l) f(n,k) - nc \right]
+ \lambda_{DE} \left[ \delta f(n,k) (p\theta h + (1-p)\theta l) - nc - (1-\delta)\gamma k \right]
+ \lambda_{DEL} \left[ (1-p)\theta l f(n,k) + p \left( \frac{\delta (p\theta h + (1-p)\theta l) f(n,k) - nc}{1-\delta} - \gamma k \right) - nc \right],
\]

which gives the first-order-conditions

\[
\frac{\partial L}{\partial n} = \frac{\delta}{1-\delta} \left[ (p\theta h + (1-p)\theta l) f_n - c \right]
+ \lambda_{DE} \left[ \delta f_n (p\theta h + (1-p)\theta l) - c \right]
+ \lambda_{DEL} \left[ (1-p)\theta l f_n + p \left( \frac{\delta (p\theta h + (1-p)\theta l) f_n - c}{1-\delta} \right) \right] = 0
\]

\[
\frac{\partial L}{\partial k} = -1 + \frac{\delta}{1-\delta} \left( p\theta h + (1-p)\theta l \right) f_k
+ \lambda_{DE} \left[ \delta f_k (p\theta h + (1-p)\theta l) - (1-\delta)\gamma \right]
+ \lambda_{DEL} \left[ (1-p)\theta l f_k + p \left( \frac{\delta (p\theta h + (1-p)\theta l) f_k - \gamma}{1-\delta} \right) \right] = 0
\]

If \( \lambda_{DE} = \lambda_{DEL} = 0 \) (which is the case for \( \delta \geq \delta(\gamma) \)), employment is at its first best and the investment level is characterized by \(-1 + \frac{\delta}{1-\delta} \left[ (p\theta h + (1-p)\theta l) f_k \right] = 0\), hence there are no overinvestments.

In a next step, assume that \( \delta < \delta(\gamma) \), i.e., \( \lambda_{DE} > 0 \) and \( \lambda_{DEL} = 0 \). This implies that \( n^* < n^{FB} \), and the investment level \( k^* \) is characterized by

\[
\left[ -1 + \frac{\delta}{1-\delta} \left( p\theta h + (1-p)\theta l \right) f_k \right] (1 + (1-\delta)\lambda_{DE}) + (1-\gamma)(1-\delta)\lambda_{DE} = 0.
\]

Hence, \(-1 + \frac{\delta}{1-\delta} \left[ (p\theta h + (1-p)\theta l) f_k \right] < 0\) for \( \gamma < 1 \) and \(-1 + \frac{\delta}{1-\delta} \left[ (p\theta h + (1-p)\theta l) f_k \right] = 0\) for \( \gamma = 1 \), and overinvestments are optimal for \( \gamma < 1 \) but not for \( \gamma = 1 \).

To compute the marginal effect of \( \gamma \) on \( k^* \), note that condition 6 yields

\[
\lambda_{DE} = -\frac{\delta}{1-\delta} \left[ \frac{(p\theta h + (1-p)\theta l) f_n - c}{\delta f_n (p\theta h + (1-p)\theta l) - c} \right],
\]

which we plug into condition 7 and obtain

\[
-\delta f_n \left( p\theta h + (1-p)\theta l \right) (1-\gamma) - \delta \left( p\theta h + (1-p)\theta l \right) f_k c + c (1-\delta \gamma) = 0
\]

Condition 8, together with the binding (DE) constraint,
\[ \delta f(n, k) \left( p\theta^h + (1-p)\theta^l \right) - nc - (1-\delta)\gamma k = 0, \]
determines effort \( n^* \) and investment \( k^* \).

This yields
\[
\frac{dk^*}{d\gamma} = \frac{-\delta (p\theta^h + (1-p)\theta^l) (fn_n (1-\gamma) + fn_k c) - \delta (fn_n (p\theta^h + (1-p)\theta^l) - c)}{\delta f_n (p\theta^h + (1-p)\theta^l) - c (1-\delta)k}
\]
where the denominator must be positive in order to satisfy the second order condition for a maximum. Hence, the sign of \( \frac{dk^*}{d\gamma} \) is determined by the sign of the nominator, which equals
\[-\delta (p\theta^h + (1-p)\theta^l) (fn_n (1-\gamma) + fn_k c) (1-\delta)k
+ \delta (fn_n (p\theta^h + (1-p)\theta^l) - c) (\delta f_n (p\theta^h + (1-p)\theta^l) - c).\]

There, the second line must be negative for the following reason: \( f_n (p\theta^h + (1-p)\theta^l) - c > 0 \) because \( n^* < n^{FB} \). \( \delta f_n (p\theta^h + (1-p)\theta^l) - c < 0 \) because otherwise, increasing \( n^* \) would relax the (DE) constraint, contradicting that it binds. The first line will in any case be positive if \( fn_k \leq 0 \), and positive or negative otherwise. Generally, the impact of positive terms is larger if \( \gamma \) is rather small, whereas the impact of the negative second term is larger if employment is rather small.

Now assume that \( \bar{\delta}(\gamma) \leq \delta < \underline{\delta}(\gamma) \), i.e., \( \lambda_{DE} = 0 \) and \( \lambda_{DEL} > 0 \). This implies that \( n^* < n^{FB} \), and the investment level \( k^* \) is characterized by
\[
\left[-1 + \frac{\delta}{1-\delta} (p\theta^h + (1-p)\theta^l) f_k\right] (1 + p\lambda_{DEL}) + \lambda_{DEL} [(1-p)\theta^l f_k + p (1-\gamma)] = 0. \]
Hence, \( -1 + \frac{\delta}{1-\delta} [(p\theta^h + (1-p)\theta^l) f_k] < 0 \) for any \( \gamma \leq 1 \), and overinvestments are optimal.

To compute the marginal effect of \( \gamma \) on \( k^* \), note that condition 6 yields
\[
\lambda_{DEL} = -\frac{\delta}{1-\delta} (p\theta^h + (1-p)\theta^l) f_n - c
\]
and obtain.
\[
-\delta p (1-\gamma) [(p\theta^h + (1-p)\theta^l) f_n - c] - (1-p)\theta^l (f_n (1-\delta) - \delta c f_k) = 0 \quad (9)
\]
Condition 9, together with the binding (DE) constraint,
\[
\delta f(n, k) (p\theta^h + (1-p)\theta^l) - nc - (1-\delta)\gamma k = 0,
\]
determines effort \( n^* \) and investment \( k^* \).

This yields
\[
\frac{dk^*}{d\gamma} = \frac{-\delta p (1 - \gamma) \left( p \theta h + (1 - p) \theta h \right) f_{nn} n - (1 - p) \theta h (f_{nn} (1 - \delta) - \delta c f_{kn}) - \delta p \left( (p \theta h + (1 - p) \theta h) f_n - c \right) \left( 1 - \delta \right) k}{\delta f_n \left( p \theta h + (1 - p) \theta h \right) - c - \delta p \left( (p \theta h + (1 - p) \theta h) f_{nk} - (1 - p) \theta h (f_{nk} (1 - \delta) - \delta c f_{kk}) \right) \delta f_k \left( p \theta h + (1 - p) \theta h \right) - (1 - \delta) \gamma},
\]

where the denominator must be positive in order to satisfy the second order condition for a maximum. Hence, the sign of \( \frac{dk^*}{d\gamma} \) is determined by the sign of the nominator, which equals
\[
\left\{ -f_{nn} \left[ \delta p (1 - \gamma) \left( p \theta h + (1 - p) \theta h \right) + (1 - p) \theta h (1 - \delta) \right] + \delta (1 - p) \theta h c f_{kn} \right\} \left( 1 - \delta \right) k
+ \delta p \left[ (p \theta h + (1 - p) \theta h) f_n - c \right] \left( \delta f_n \left( p \theta h + (1 - p) \theta h \right) - c \right).
\]

There, the second line might be positive or negative. The first line will in any case be positive if \( f_{nk} \leq 0 \), and positive or negative otherwise. ■

C Optimal Firm Behavior Given Cash Holdings are Possible

Starting from the equilibrium we derived in Section 4, assume that whenever demand conditions are high, the firm can retain some of its earnings. These cash reserves are used to increase employment from the next period on and compensate the additional agents. After a number of subsequent low-demand periods, the cash reserves are used up and employment is down at its original level \( n^* \) - until the next high-demand period. In the following, we analyze to what extent retaining earnings is optimal for the firm.

Define \( m \geq 1 \) as the number of periods a higher employment level can at least be enforced, i.e., \( m \) is the subsequent number of low-demand periods after which all cash reserves are used up. Furthermore, define the amount retained as \( s_m \), and the employment level in the first period after \( s_m \) has been accumulated as \( n_m \).

Now, assume that \( s_m \) has been retained and employment raised to \( n_m \) in the following period. If the firm faces a low-demand shock in this period, all \( n_m \) agents are compensated accordingly. However, some of the retained cash is needed, and available funds go down to \( s_{m-1} \). Furthermore, employment in the next period will be \( n_{m-1} \). This process is continued until either all cash reserves are used (and employment is at \( n_0 = n^* \)) or a high-demand shock allows to fill up cash reserves and increase employment to \( n_m \) again. To keep the problem tractable, we assume that income in a high-demand state is sufficiently large such that the optimal amount \( s_m \) can be retained in one high-demand state, i.e., we impose

Assumption A1: Assume the firm can retain earnings. Then, (Lh) does not bind in a profit-maximizing equilibrium.
Assumption A1 implies that only one high-demand state is needed in order to fill up the firm’s desired cash stock.

**Profits**

We write payoffs as functions of the remaining subsequent low-demand shocks before all cash reserves are used up. Profits given retained earnings are at its maximum level are denoted \( \tilde{\Pi}(m) \), since \( m \) has been defined as the number of periods an employment level above \( n^* \) can at least be enforced. Hence,

\[
\Pi(m) = p \left[ \theta^h f(n_m, k) - n_m b_m^h + \delta \Pi(m) \right] \\
+ (1 - p) \left[ \theta^l f(n_m, k) - n_m b_m^l + (s_m - s_{m-1}) + \delta \Pi(m-1) \right].
\]

Note that bonus payments are the amounts actually paid out to agents. Therefore, the reduction of cash holdings in a low-demand state enters the principal’s profits positively.

Furthermore, profits after \( j < m \) subsequent low-demand periods are

\[
\Pi(m - j) = p \left[ \theta^h f(n_{m-j}, k) - n_{m-j} b_{m-j}^h - (s_m - s_{m-j}) + \delta \Pi(m) \right] \\
+ (1 - p) \left[ \theta^l f(n_{m-j}, k) - n_{m-j} b_{m-j}^l + (s_{m-j} - s_{m-j-1}) + \delta \Pi(m - j - 1) \right].
\]

After \( m - 1 \) subsequent low-demand period, a higher employment can only be enforced for maximum one more low-demand period. Then,

\[
\Pi(1) = p \left[ \theta^h f(n_1, k) - n_1 b_1^h - (s_m - s_1) + \delta \Pi(m) \right] \\
+ (1 - p) \left[ \theta^l f(n_1, k) - n_1 b_1^l + s_1 + \delta \Pi(0) \right].
\]

Finally,

\[
\Pi(0) = p \left( f(n, k) \theta^h - n_0 b^h - s_m + \delta \Pi(m) \right) \\
+ (1 - p) \left( f(n, k) \theta^l - n_0 b^l + \delta \Pi(0) \right)
\]

are profits given the firm has used all its cash.

**Objective**

The objective is to find levels of \( m \geq 0 \), \( n_{m-j} \) (\( j \leq m \)) and the respective amounts of cash that maximize \( \Pi(0) \), given the constraints derived below. Hence, we maximize profits given no cash is initially available. It will turn out, though, that the respective strategy also maximizes \( -s_m + \delta \Pi(m) \), the firm’s objective given savings could also be raised at the beginning of the game (when capital \( k \) is raised).

**Constraints**

The following constraints have to be satisfied. For all \( j \in [0, m] \), dynamic enforcement constraints for low- and high-demand states must hold:
\[ n_{m-j}b^l_{m-j} \leq \delta \Pi(m-j) - s_{m-j} \quad \text{(DEl\text{j})} \]

\[ n_{m-j}b^h_{m-j} \leq \delta \Pi(m) - s_m. \quad \text{(DEh\text{j})} \]

Cash holding enter the above constraints since if the principal reneges on payments promised to agents, it will also be optimal to consume accumulated savings. Furthermore, note that \( s_0 = 0 \).

Since (Lh) is satisfied by assumption, liquidity constraints must only hold for low-demand states.

For all \( j \in [0, m-1] \), we have

\[ n_{m-j}b^l_{m-j} \leq \theta^l f(n_{m-j}, k) + (s_{m-j} - s_{m-j-1}), \quad \text{(Ll\text{j})} \]

where \( s_{-1} = 0 \) in LLl(0).

In addition (IC) constraints must hold (where we already take into account that agents receive no rent), namely

\[ pb^h_{m-j} + (1 - p)b^l_{m-j} - c \geq 0 \]

for all \( j \in [0, m] \).

For the same reasons as before, (IC) constraints will bind on the equilibrium path, giving

\[ b^l_{m-j} = \frac{c - pb^h_{m-j}}{1 - p}. \]

**Results**

In this section, we first assume that it is optimal to accumulate strictly positive cash reserves (which implies \( m \geq 1 \)) and derive properties of a profit-maximizing equilibrium. Then, we work out conditions under which it is actually optimal to retain earnings.

First, we show that given retaining earnings is optimal, (DEl) constraints can be omitted.

**Lemma 5** Assume \( m \geq 1 \). Then, all (DEl) are automatically implied by the respective (Ll) constraints.

**Proof of Lemma 5:** First, we plug \( b^l_{m-j} = \frac{c - pb^h_{m-j}}{1 - p} \) into profits, which yields the set of constraints

\[ n_{m-j} \frac{c - pb^h_{m-j}}{1 - p} \leq \delta \Pi(m-j) - s_{m-j} \quad \text{(DEl\text{j})} \]
\[ n_{m-j}b_{m-j}^h \leq \delta \Pi(m) - s_m \quad \text{(DEh}(j)) \]

\[ n_{m-j} \frac{c - pb_{m-j}^h}{1 - p} \leq \theta^f(n_{m-j}, k) + (s_{m-j} - s_{m-j-1}) \quad \text{(Ll}(j)) \]

Since the left hand sides of DEL(j) and Ll(j) constraints are the same, those do generally not bind simultaneously. Put differently, if \( \delta \Pi(m-j) - s_{m-j} \geq \theta^f(n_{m-j}, k) + (s_{m-j} - s_{m-j-1}) \), DEL(j) constraints are implied by Ll(j) constraints.

To the contrary, assume there is a \( j^* \geq 0 \) where \( \delta \Pi(m-j^*) - s_{m-j^*} < \theta^f(n_{m-j^*}, k) + (s_{m-j^*} - s_{m-j^*-1}) \).

Reduce all \( s_{m-j} \) with \( j \leq j^* \) by a small \( \varepsilon > 0 \). This tightens Ll\((j^*)\) which however will still hold for \( \varepsilon \) sufficiently small. Ll\((j)\) constraints for all \( j \) besides \( j^* \) remain unaffected. Furthermore, profits change by \( \Delta \Pi(0) = p\varepsilon \frac{1 - (\delta(1-p))^{j^*+1}}{1 - \delta(1-p)} \), hence go up.

It remains to show that no other constraint is violated by this operation. There, note that

\[ \Delta \Pi(m) = -\varepsilon (\delta(1-p))^{j^*} (1-p) \]

\[ \Delta \Pi(m - j^*) = -\frac{\varepsilon (1-p)}{1 - \delta(1-p)} \left( 1 - \delta + \delta p (\delta(1-p))^{j^*} \right) \]

and for any \( 1 < k < m \),

\[ \Delta \Pi(m - (j^* - k)) = \frac{\delta p \Delta \Pi(m) - \varepsilon (\delta(1-p))^k [1 - p] (1 - \delta)}{1 - \delta(1-p)} \]

and

\[ \Delta \Pi(m - (j^* + k)) = \frac{p\varepsilon \left( 1 - (\delta(1-p))^{j^*+1} \right)}{1 - \delta(1-p)}. \]

Now we can show that no constraint is violated if all \( s_{m-j} \) with \( j \leq j^* \) are reduced by a \( \varepsilon > 0 \):  

- DEh\((j)\) are relaxed because each right hand side changes by \( \varepsilon \left( 1 - (\delta(1-p))^{j^*+1} \right) > 0 \)  
- DEL\((0)\) is relaxed because its right hand side changes by \( \varepsilon \left( 1 - (\delta(1-p))^{j^*+1} \right) > 0 \)  
- DEL\((j^*)\) is relaxed, because its right hand side changes by \( \varepsilon \left( \frac{(1-\delta)(1-\delta(1-p)) + \delta p (1-(\delta(1-p))^{j^*+1})}{1-\delta(1-p)} \right) > 0 \)
DE\((j^* + k)\) are relaxed because each right hand side changes by \(\frac{p\delta(1-(\delta(1-p))^{j^*+1})}{1-\delta(1-p)} > 0\).

DE\((j^* - k)\) are relaxed because each right hand side changes by
\[\varepsilon(1-\delta(1-(\delta(1-p)))^{k+1}) + \delta p(1-(\delta(1-p))^{j^*+1}) > 0\]

Hence, reducing all \(s_{m-j}\) with \(j \leq j^*\) by a small \(\varepsilon > 0\) does not violate any constraint but increases profits.

Lemma 5 implies that all DEl constraints can be omitted given holding cash is optimal. This allows us to simplify the problem by adding DEh\((j)\) (multiplied with \(p\)) and LLl\((j)\) (multiplied with \((1-p))\) constraints for each \(j\), which gives a set of necessary and sufficient constraints (sufficiency follows from the same reasoning as in the situation without savings):

\[
\delta p\tilde{\Pi}(m) - ps_m + (1-p)\theta^h f(n_{m-j}, k) - n_{m-j}c + (1-p)(s_{m-j} - s_{m-j-1}) \geq 0, \quad (DE-L(j))
\]

where \(s_0 = s_{-1} = 0\).

In a next step, we show that given holding cash is optimal, all DE-L constraints must bind:

**Lemma 6** If \(s_m > 0\), \(DE-L(j)\) constraints bind for each \(j \in \{0,...,m\}\).

**Proof of Lemma 6:** Assume there is a \(j^*\) where DE-L\((j^*)\) did not bind. Then, the respective employment level can be raised without violating any constraint, thereby increasing profits.

Lemma 6 allows us to plug the binding DE-L constraints
\[
(\delta p\tilde{\Pi}(m) = n_0 c - (1-p)\theta^h f(n_0, k) + ps_m\text{ and}\]
\[
(s_{m-j} - s_{m-j-1}) = \frac{n_{m-j}c - \delta p\tilde{\Pi}(m) - (1-p)\theta^h f(n_{m-j}, k) + ps_m}{(1-p)}\]

into profits, giving
\[
\Pi(0) = \frac{p}{1-\delta(1-p)} f(n_0, k)\theta^h
\]
\[
\Pi(m-j) = p\theta^h f(n_{m-j}, k) + ps_{m-j} + \delta(1-p)\Pi(m-j-1),
\]
and
\[
\Pi(m) = \sum_{i=0}^{m-1} (\delta(1-p))^i \left(p\theta^h f(n_{m-i}, k) + ps_{m-i}\right) + (\delta(1-p))^m \frac{p}{1-\delta(1-p)} f(n_0, k)\theta^h
\]
In a next step, we can show that employment is gradually going down once cash reserves are used in low-demand periods:

**Lemma 7** \( n_{m-j+1} > n_{m-j} \) for all \( j \in [1,m] \)

**Proof of Lemma 7:** First, we show that \( n_{m-j+1} \geq n_{m-j} \). To the contrary, assume there is a \( \tilde{j} \) with \( n_{m-j+1} < n_{m-j} \). Replace both employment levels and reduce \( s_{m-j} \) to keep DE-L(\( \tilde{j} + 1 \)) unaffected, i.e.,

\[
\Delta s_{m-j} = \theta^j f(n_{m-j}, k) - \frac{n_{m-j} c}{(1 - p)} - \left( \theta^j f(n_{m-j+1}, k) - \frac{n_{m-j+1} c}{(1 - p)} \right).
\]

Note that this leaves DE-L(\( \tilde{j} \)) as well as other constraints unaffected. However, this change increases \( \Pi(m) \) and thereby \( \Pi(0) \):

\[
\frac{\Delta \Pi(m)}{(\delta(1-p))^{j-1}} = p^{\theta^h} (f(n_{m-j}, k) - f(n_{m-j+1}, k)) + \delta(1-p)p^{\theta^h} (f(n_{m-j+1}, k) - f(n_{m-j}, k))
\]

\[
+ p \left[ \theta^j f(n_{m-j}, k) - \frac{n_{m-j} c}{(1 - p)} - \left( \theta^j f(n_{m-j+1}, k) - \frac{n_{m-j+1} c}{(1 - p)} \right) \right]
\]

\[
\geq p^{\theta^h} (f(n_{m-j}, k) - f(n_{m-j+1}, k)) + (1-p) p^{\theta^h} (f(n_{m-j+1}, k) - f(n_{m-j}, k))
\]

\[
+ p \left[ \theta^j f(n_{m-j}, k) - \frac{n_{m-j} c}{(1 - p)} - \left( \theta^j f(n_{m-j+1}, k) - \frac{n_{m-j+1} c}{(1 - p)} \right) \right]
\]

\[
= p^{\theta^h} (f(n_{m-j}, k) + (1-p)\theta^j f(n_{m-j}, k) - n_{m-j} c)
\]

\[
- p \left( p^{\theta^h} f(n_{m-j+1}, k) + (1-p)\theta^j f(n_{m-j+1}, k) - n_{m-j+1} c \right) > 0,
\]

where the last inequality follows from \( n_{m-j} > n_{m-j+1} \), and from both employment levels being inefficiently low.

To complete the proof, it remains to show that \( n_{m-j+1} = n_{m-j} \) is not possible. To the contrary, assume there is a \( \hat{j} \) with \( n_{m-j+1} = n_{m-j} \). Now, marginally increase \( n_{m-j+1} \) and marginally decrease \( n_{m-j} \). Furthermore, have \( ds_{m-j} = \theta^j \frac{\partial f(n_{m-j+1}, k)}{\partial n} - \frac{c}{1-p} \) to keep DE-L(\( \hat{j} + 1 \)) and DE-L(\( \hat{j} \)) unaffected. Then,

\[
\frac{d \Pi(m)}{(\delta(1-p))^{j-1}} = p^{\theta^h} \frac{\partial f(n_{m-j+1}, k)}{\partial n} + \delta(1-p) \left( -p^{\theta^h} \frac{\partial f(n_{m-j+1}, k)}{\partial n} + p \left( \theta^j \frac{\partial f(n_{m-j+1}, k)}{\partial n} - \frac{c}{1-p} \right) \right)
\]

\[
\geq p \left( p^{\theta^h} \frac{\partial f(n_{m-j+1}, k)}{\partial n} + (1-p)\theta^j \frac{\partial f(n_{m-j+1}, k)}{\partial n} - c \right) > 0.
\]

Finally, we can show that maximum employment \( n_m \) is independent of \( m \).
Lemma 8 Maximum employment $n_m$ is characterized by

$$\frac{\delta (1-p)}{(1-\delta p)} \rho\delta f(n_m,k) \frac{\partial f(n_m,k)}{\partial n_m} + (1-p)\theta \frac{\partial f(n_m,k)}{\partial n_m} - c = 0.$$ 

Proof of Lemma 8: Note that if holding cash is optimal, all DE-L(j) constraints bind for a given $m$, which implies that

$$\delta p \Pi(m) - ps_m + (1-p)\theta^j f(n_{m-j}, k) - n_{m-j}c + (1-p) \left(s_{m-j} - s_{m-j-1}\right) = 0$$

can be used to obtain the necessary savings for all levels of $n_{m-j}$. Then, the objective is to maximize $\Pi(0) = \frac{p}{1-\delta(1-p)} f(n_0, k)\theta^h$. Thus, employment levels $n_{m-j}$ are chosen in order to maximize $n_0$. Holding cash can only be optimal if DE-L(0) binds, in which case $n_0$ is determined by $\delta p \Pi(m) - ps_m + (1-p)\theta^j f(n_0, k) - n_0c = 0$. Hence, employment levels $n_{m-j}$ are optimally chosen to maximize $\delta \Pi(m) - s_m$ and determined by setting $\frac{\partial (\delta \Pi(m) - s_m)}{\partial n_{m-j}} = 0$.

This substantially simplifies the analysis, since generally all savings and employment levels are interconnected.

Furthermore, $\frac{\partial s_{m-j}}{\partial n_{m-j}} = 0$ for $\hat{j} > \hat{j}$, i.e. in particular $\frac{\partial s_{m-j}}{\partial n_{m-j}} = 0$ for $j > 0$. This is implied by solving DE-L(m-j) for savings level $s_{m-j}$:

$$s_{m-j} = \sum_{i=0}^{j} \frac{\delta p \Pi(m) - ps_m + (1-p)\theta^j f(n_{m-i}, k) - n_{m-i}c}{1-p},$$

and using $\frac{\partial (\delta \Pi(m) - s_m)}{\partial n_{m-j}} = 0$.

Hence, solving for DE-L(m) for $s_m$ and plugging the value into $\delta \Pi(m) - s_m$ gives

$$\frac{\partial (\delta \Pi(m) - s_m)}{\partial n_m} = \delta p \rho^h \frac{\partial f(n_m,k)}{\partial n_m} - \frac{\delta p^2 \rho^h \frac{\partial f(n_m,k)}{\partial n_m} + (1-p)\theta^j \frac{\partial f(n_m,k)}{\partial n_m} - c}{(1-2\rho^j + \rho^{2j})} (1-\delta p),$$

and $n_m$ is determined by

$$\frac{\delta (1-p)}{(1-\delta p)} \rho^h \frac{\partial f(n_m,k)}{\partial n_m} + (1-p)\theta^j \frac{\partial f(n_m,k)}{\partial n_m} - c = 0.$$

This result shows that a larger value of $m$ does not increase maximum employment, but rather “smoothes” the process of layoffs after negative demand shocks.

Finally, we can prove Proposition 3:

Proof of Proposition 5: Lemma (8) gives (4) for maximum employment. Furthermore, note that if savings are optimal, then $n_m > n_{m-1} > ... > n$. Hence, if maximum enforceable employment without cash reserves satisfies $\left(\frac{\delta (1-p)}{(1-\delta p)} \rho^h + (1-p)\theta^j\right) f_n - c \leq 0$, holding cash cannot be optimal.

Here, we do not aim for solving for the optimal $m$, i.e. the maximum number of subsequent negative shocks until cash reserves are used up. The optimal level of $m$ would again be determined by maximizing $\delta \Pi(m) - s_m$. To get around integer problems, we would first treat
$m$ as a continuous variable, set \( \frac{\partial (\delta \Pi (m) - s_m)}{\partial m} = 0 \) and get the optimal $m$ as the largest integer for which \( \frac{\partial (\delta \Pi (m) - s_m)}{\partial m} \geq 0 \).
References


